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USING A STOCHASTIC MODEL IN STRATIFIED SAMPLING

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College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

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Summary:

The objective of this paper is to explore the effects of increasing the amount of monetary error when the audited amounts are generated by a stochastic model. The increased variance when there is a material monetary error makes it desirable to design the sample in a manner different from current practice.

Using a Stochastic Model in Stratified Sampling

Introduction. To use stratified sampling techniques to test for the potential existence of a material monetary error, the auditor must select a basis for stratification and determine an appropriate sample size. The auditor's objective is to select a sample size that will maintain the sampling risk (β) and risk of overauditing (α) at tolerable levels. Accomplishing this objective requires knowledge of the sampling distribution of $\hat{X}$, the estimated total audited amount, under two conditions: when the population of recorded amounts contains no monetary error and when total monetary error equals M, a pre-determined material amount. A complicating factor is that as the amount of monetary error increases, the standard deviation of audited amounts (or difference amounts) may change. The purpose of this paper is to present a planning methodology that takes this complication into account.

Model. A useful way of studying the effects of monetary errors on the sampling distribution of the estimated total audited amount is to use a superpopulation model. From a superpopulation viewpoint the observed audited amounts are assumed to be the realized outcomes of a prescribed random process. Superpopulation models have a long history in the sampling literature. Early users are Cochran (1939, 1946), Deming and Stephan (1941) and Madow and Madow (1944).

A model for the audited amounts in the population is

$$\underset{\sim}{X}_j = (1 - \underset{\sim}{\theta}_j)Y_j , \qquad j = 1, \ldots, N.$$

In this model, the recorded amount (Y_j) is not a random variable, but the associated audited amount (X_j) is a realization of a random process. The random variable $\underset{\sim}{X}_j$ is generated from the recorded amount Y_j by multiplication by the factor $(1 - \underset{\sim}{\theta}_j)$. $\underset{\sim}{\theta}_j$ is a random variable which takes on the value zero (0) with probability $(1 - \pi)$, and with probability π, takes a value in the support of F, a distribution function.

Kaplan (1973A) used a similar model, but he seems to have regarded the recorded amounts as also being random variables. If the support of F is the interval (0,1], $\underset{\sim}{\theta}_j$ represents the relative amount of overstatement associated with the recorded amount Y_j. If F is a jump function with a single jump at one (1), the audited amount $\underset{\sim}{X}_j$ is Y_j with probability $(1 - \pi)$ and zero (0) with probability π.

Using the notation that E_θ represents the expectation operator with respect to the random variable $\underset{\sim}{\theta}$, the following relationships hold:

(1) $$E_\theta \underset{\sim}{X}_j = Y_j(1 - \pi\mu_\theta) \text{ where } \mu_\theta = \int \theta dF(\theta)$$

and

(2) $$E_\theta \mathrm{Var}\underset{\sim}{X}_j \doteq (1 - \pi\mu_\theta + \pi\sigma_\theta^2)\mathrm{Var}Y + (\pi\mu_\theta(1 - \pi\mu_\theta) + \pi\sigma_\theta^2)\bar{Y}^2$$

where $\mathrm{Var}\,\underset{\sim}{X} = \dfrac{\Sigma(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})^2}{N}$, $\sigma_\theta^2 = \int(\theta - \mu_\theta)^2 dF(\theta)$, and the symbol $\doteq$ means approximate equality. The approximation is obtained by substituting one (1) for the quantity $(\frac{N - 1}{N})$, and so the terms on the right somewhat overstate the expected variance.

The above relationship shows that the expected variance of audited amounts exceeds the variance of recorded amounts whenever

$$(3) \qquad CV^2(Y) < \frac{\pi\mu_\theta(1 - \pi\mu_\theta) + \pi\sigma_\theta^2}{\pi(\mu_\theta - \sigma_\theta^2)} = \frac{\mu_\theta(1 - \pi\mu_\theta) + \sigma_\theta^2}{(\mu_\theta - \sigma_\theta^2)},$$

where $CV^2(Y)$ denotes the square of the coefficient of variation of the recorded amounts (Y_j). The coefficient of variation equals the ratio of the standard deviation to the mean.

As a special case, because

$$(1 - \pi) < \frac{\mu_\theta(1 - \pi\mu_\theta) + \sigma_\theta^2}{(\mu_\theta - \sigma_\theta^2)},$$

when the support of F is the interval (0,1] (all monetary errors are errors of overstatement), it follows that the expected variance of audited amounts exceeds the variance of recorded amounts whenever

$$(4) \qquad CV^2(Y) < (1 - \pi).$$

A further result that easily follows from (2) is that for π fixed, the expected variance of audited amounts is least among all distribution functions F with a fixed mean, μ_θ, when $\sigma_\theta^2 = 0$. That is, for a fixed mean and fixed proportion of recorded amounts having monetary error, the distribution of relative error that concentrates at μ_θ yields the smallest expected variance.

Finally, whenever $\pi < \frac{1}{2}$ and monetary errors represent overstatements, the expected variance of recorded amounts is least among all distributions concentrated at a single point when that point is one (1).

Similar results hold for the difference amounts

$$\underset{\sim}{D}_j = Y_j - \underset{\sim}{X}_j$$

$$= \underset{\sim}{\theta}_j Y_j.$$

Specifically,

$$E_\theta \underset{\sim}{D}_j = \pi \mu_\theta Y_j$$

and

$$E_\theta \mathrm{Var}\underset{\sim}{D} \doteq \pi(\mu_\theta^2 + \sigma_\theta^2)\mathrm{Var}Y + (\pi(1 - \pi)\mu_\theta^2 + \pi\sigma_\theta^2)\bar{Y}^2 \tag{5}$$

From (5) it follows that the expected variance of the monetary errors exceeds the variance of the recorded amounts whenever

$$CV^2(Y) < \frac{\pi\sigma_\theta^2 + \pi(1 - \pi)\mu_\theta^2}{1 - \pi(\mu_\theta^2 + \sigma_\theta^2)}.$$

Because the right-hand side of (5) becomes larger when σ_θ^2 is made larger, we can replace σ_θ^2 by $\mu_\theta(1 - \mu_\theta)$ to obtain the following inequality provided that all monetary errors represent overstatement.

$$E_\theta \mathrm{Var}\underset{\sim}{D} \leq (\pi\mu_\theta)\mathrm{Var}Y + \pi\mu_\theta(1 - \pi\mu_\theta)\bar{Y}^2 \tag{6}$$

From (6) it follows that the variance of the recorded amounts will exceed the expected variance of monetary errors when

$$CV^2(Y) > \pi\mu_\theta.$$

Application. Currently, many auditors plan the extent of a statistical substantive test by analyzing the recorded amounts. A common procedure is

to form L strata using the CUMF technique, determine the sample size using the standard deviations of the stratum recorded amounts, and allocate the total sample size to the strata using Neyman allocation.

For this procedure to produce a sample size that is appropriate for the purposes of maintaining the sampling risk (β) and the risk of over-auditing (α) at prescribed tolerable levels, the standard deviation of recorded amounts in each stratum must be sufficiently large. The requisite size, of course, depends upon the statistical technique involved.

Conceptually, the auditor may formulate an hypothesis that the total monetary error, D, equals (or exceeds) a material amount, M, and test this hypothesis against the alternative that the total monetary error is less than a material amount. Specializing this to a test for overstatement errors, the following representation obtains:

$$H_0: \quad D \geq M$$
$$H_1: \quad D < M,$$

where D represents the total overstatement error and M represents a material amount of monetary error (note that $D = Y - X$, where Y represents the total recorded amount and X the total audited amount).

To test this hypothesis the recorded amounts may be divided into L strata, a stratified random sample selected and evaluated to produce an estimated audited amount, $\hat{X}$, or equivalently, an estimated total error amount $\hat{D} = Y - \hat{X}$. The auditor determines the sample size so that his risks are within tolerable levels. Specifically, a decision rule is desired with the properties that

$$\Pr\{\text{decide there is no material error} \mid D = M\} = \beta$$

and

$$\Pr\{\text{decide there may be material error} \mid D = 0\} = \alpha$$

where β and α are the specified tolerable levels.[1]

Finding an appropriate decision rule involves knowing the sampling distribution of $\hat{X}$ (or $\hat{D}$) both when $D = M$ and when $D = 0$. If the sampling distribution is degenerate at $D = 0$, then $D = E$, where E represents an insignificant monetary error, may be used in place of $D = 0$.

A difficulty that has been observed is that the standard error of $\hat{X}$ (or $\hat{D}$) depends upon the magnitude of the monetary error present in the population. The symbol $\sigma_D(\hat{X})$ (or $\sigma_D(\hat{D})$) will be used to denote this standard error. If this standard error were known, and if the sampling distribution of $\hat{X}$ (or $\hat{D}$) is approximately normal, the decision rule and sample size could be determined as follows:

Decide there is no material error whenever

$$Y - \hat{X} + z_\beta \sigma_M(\hat{X}) < M$$

and whenever this is not satisfied, decide that there may be a material amount of monetary error.

Select the sample size so that

$$\Pr\{Y - \hat{X} + z_\beta \sigma_M(\hat{X}) \geq M \mid D = 0\} = \alpha.$$

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[1]The use of the symbols β and α conforms to common usage in auditing.

Using the assumed normality of the sampling distribution, this will be satisfied whenever

$$M = \sigma_M(\hat{X})[z_\beta + z_\alpha \cdot \frac{\sigma_0(\hat{X})}{\sigma_M(\hat{X})}].$$

whenever $\sigma_0(\hat{X}) \geq \sigma_M(\hat{X})$, the sample size could be conservatively determined so that

$$\frac{M}{z_\beta + z_\alpha} = \sigma_0(\hat{X}), \text{ and}$$

whenever $\sigma_M(\hat{X}) \geq \sigma_0(\hat{X})$, the sample size could be conservatively determined so that

$$\frac{M}{z_\beta + z_\alpha} = \sigma_M(\hat{X}).$$

Alternatively, the decision rule could be stated directly in terms of $\hat{D}$ and the corresponding $\sigma_D(\hat{D})$. If the sampling distribution is degenerate for D = 0, then the condition D = 0 may be replaced by D = E.

Using a stratified mean estimate ($\hat{X}_{MS}$) of the total audited amount, it is required that the stratum standard deviation of recorded amounts be at least as large as the stratum standard deviation of audited amounts when the total monetary error is material in amount. When this is not so, the actual sampling risk may exceed the nominal sampling risk.

The fact that the actual sampling risk often exceeds the nominal sampling risk has been observed by many authors. The reason most often cited for this is that too few errors are observed in the sample (see Neter and Loebbecke (1975). Kaplan (1973B) offers the explanation that there is correlation

between the magnitude of the estimate of total monetary error and the estimate of the standard error. The explanation offered here is that one reason for underestimating the standard error is that the sample size may be too small and not appropriately allocated to the strata.

To illustrate this, suppose that a population consisting of 10,000 items has a total recorded amount of $1,000,000. Dividing these items into two strata based on the recorded amounts might produce the following results:

	Population Size	Mean Recorded Amount	Standard Deviation of Recorded Amounts	Square of Coefficient of Variation
Stratum 1	6000	50	13	.0676
Stratum 2	4000	175	20	.0131

If the auditor specifies a material amount M = $50,000 and $\alpha = \beta = .05$, the required sample size based on the recorded amounts is 129. Optimum allocation to the strata gives 64 to stratum 1 and 65 to stratum 2.

Assuming that all monetary errors are overstatements, and that the monetary errors are randomly distributed among the 10,000 items, the smallest expected variance of recorded amounts occurs when .05 of the items are 100 percent overstated. Using equation (2), it follows that the expected standard deviation of audited amounts in stratum 1 is 16.71 and in stratum 2 is 42.83. Using these values, the required sample size is 368, with 136 allocated to stratum 1 and 232 to stratum 2.

In this example both the total sample size of 129 is too small and the relative amount of the sample allocated to stratum 2 is much smaller than warranted.

In practice, of course, the auditor does not know the value of the standard error of the estimate when there is a material amount of monetary error in the population. Nevertheless, certain courses of action are possible

1. Design the sampling plan so that $\sigma_0(\hat{\bar{X}})$ will be at least as large as $\sigma_M(\hat{\bar{X}})$ (or $\sigma_M(\hat{D})$).
2. Design the sampling based on an estimate of $\sigma_M(\hat{\bar{X}})$ (or $\sigma_M(\hat{D})$).

To implement either suggested action the auditor needs to consider the possible distribution of monetary error supposing the total monetary error equals a material amount. From equation (1), it follows that if monetary errors are randomly distributed among the population items,

$$(7) \qquad \pi\mu_\theta = \frac{M}{Y}$$

This relationship implies that the proportion of items in error may range from $\frac{M}{Y}$ up to 100 percent.

To study the effect of the distribution of θ, the following class of distributions is considered for the case of overstatement errors.

$$f(\theta) = c\theta^{c-1}, \qquad 0 < \theta \leq 1,\ c > 0.$$

For this class of distributions, $\mu_\theta = \frac{c}{c+1}$ and $\sigma_\theta^2 = \frac{c}{(c+1)^2(c+2)}$. The shape of the distribution is determined by the parameter c. In the range $0 < c < 1$, the density function decreases monotomically over the interval $0 < \theta \leq 1$. For example, Figure 1 is a plot of the density when $c = \frac{1}{2}$.

Figure 1 about here

$$f(\theta) = \frac{1}{2}\theta^{-\frac{1}{2}} \quad 0 < \theta \leq 1$$

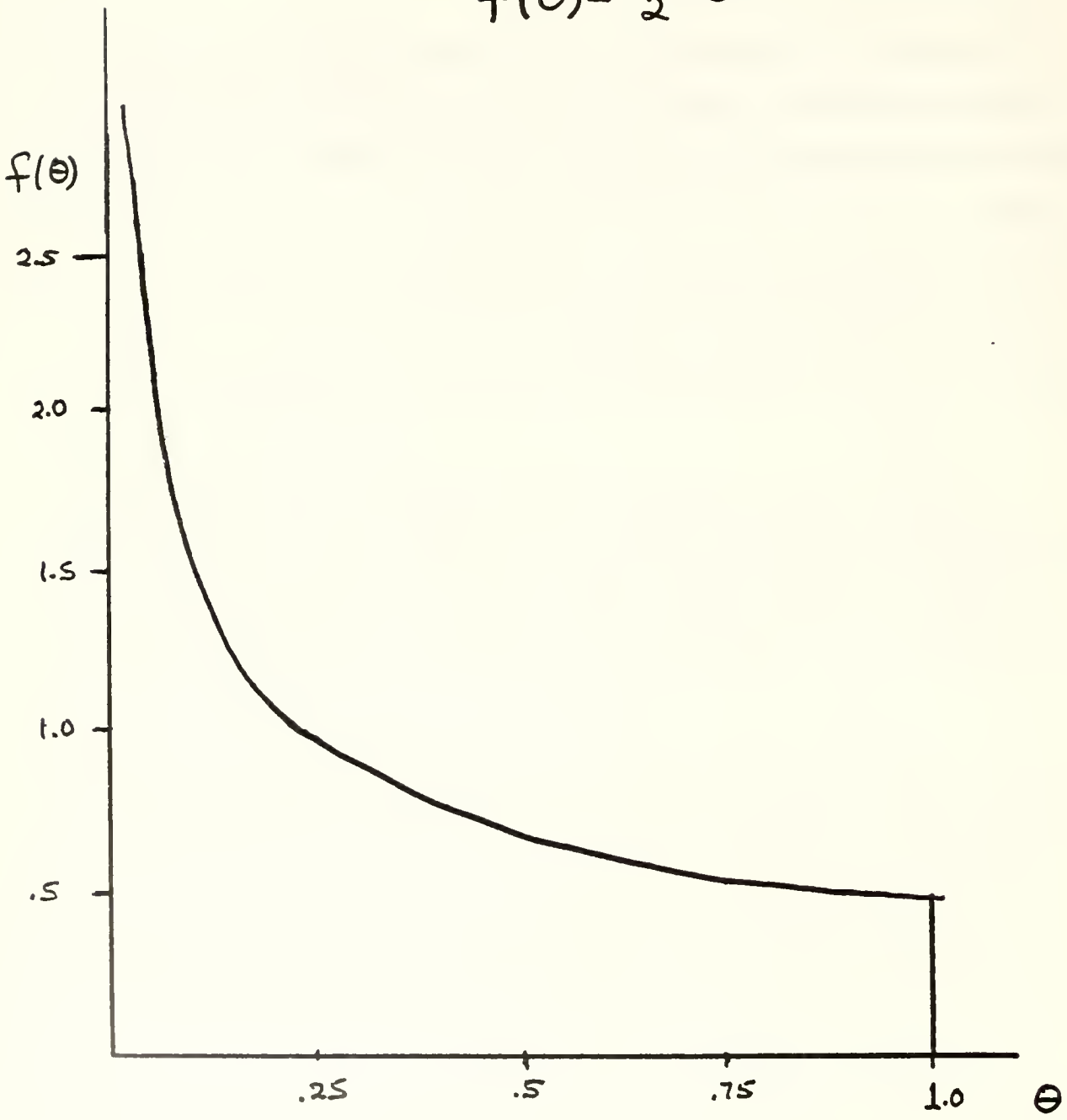

FIGURE 1

When $c = 1$, the distribution is uniform over the interval $0 < \theta \leq 1$, and when $c > 1$, the density increases over the interval $0 < \theta \leq 1$. Figure 2 is a plot of the density for $c = 3$.

Figure 2 about here

In the limiting case ($c \to +\infty$), the distribution is entirely concentrated at $\theta = 1$. This corresponds to having all monetary errors be 100 percent overstatement errors.

From (2), the expected variance of the audited amounts is given by the following expression

$$(8) \qquad E_\theta \underset{\sim}{\mathrm{Var}X} = \left(1 - \pi\frac{c}{c+1} + \pi\frac{c}{(c+1)^2(c+2)}\right)\mathrm{Var}\ Y + \left(\pi\frac{c}{c+1}\left(1 - \pi\frac{c}{c+1}\right) + \pi\frac{c}{(c+1)^2(c+2)}\right)\bar{Y}^2.$$

To obtain the value of this expected variance when there is a material amount of monetary error, we set $\pi\frac{c}{c+1} = \frac{M}{Y}$. This yields

$$(9) \qquad E_\theta \underset{\sim}{\mathrm{Var}X} = \left(1 - \frac{M}{Y} + \frac{\frac{M}{Y}\left(1 - \frac{1}{\pi}\frac{M}{Y}\right)^2}{\left(2 - \frac{1}{\pi}\frac{M}{Y}\right)}\right)\mathrm{Var}Y + \left(\frac{M}{Y}\left(1 - \frac{M}{Y}\right) + \frac{\frac{M}{Y}\left(1 - \frac{1}{\pi}\frac{M}{Y}\right)^2}{\left(2 - \frac{1}{\pi}\frac{M}{Y}\right)}\right)\bar{Y}^2$$

The expected variance increases as π increases from its minimum value of $\frac{M}{Y}$ to 1. At $\pi = 1$, the expected variance is

$$(10) \qquad E_\theta \underset{\sim}{\mathrm{Var}X} = \left(1 - \frac{M}{Y}\right)\left(1 + \frac{\frac{M}{Y}\left(1 - \frac{M}{Y}\right)}{2 - \frac{M}{Y}}\right)\mathrm{Var}Y + \frac{M}{Y}\left(1 - \frac{M}{Y}\right)\left(1 + \frac{\left(1 - \frac{M}{Y}\right)}{2 - \frac{M}{Y}}\right)\bar{Y}^2$$

while at $\pi = \frac{M}{Y}$,

$$(11) \qquad E_\theta \underset{\sim}{\mathrm{Var}X} = \left(1 - \frac{M}{Y}\right)\mathrm{Var}Y + \frac{M}{Y}\left(1 - \frac{M}{Y}\right)\bar{Y}^2$$

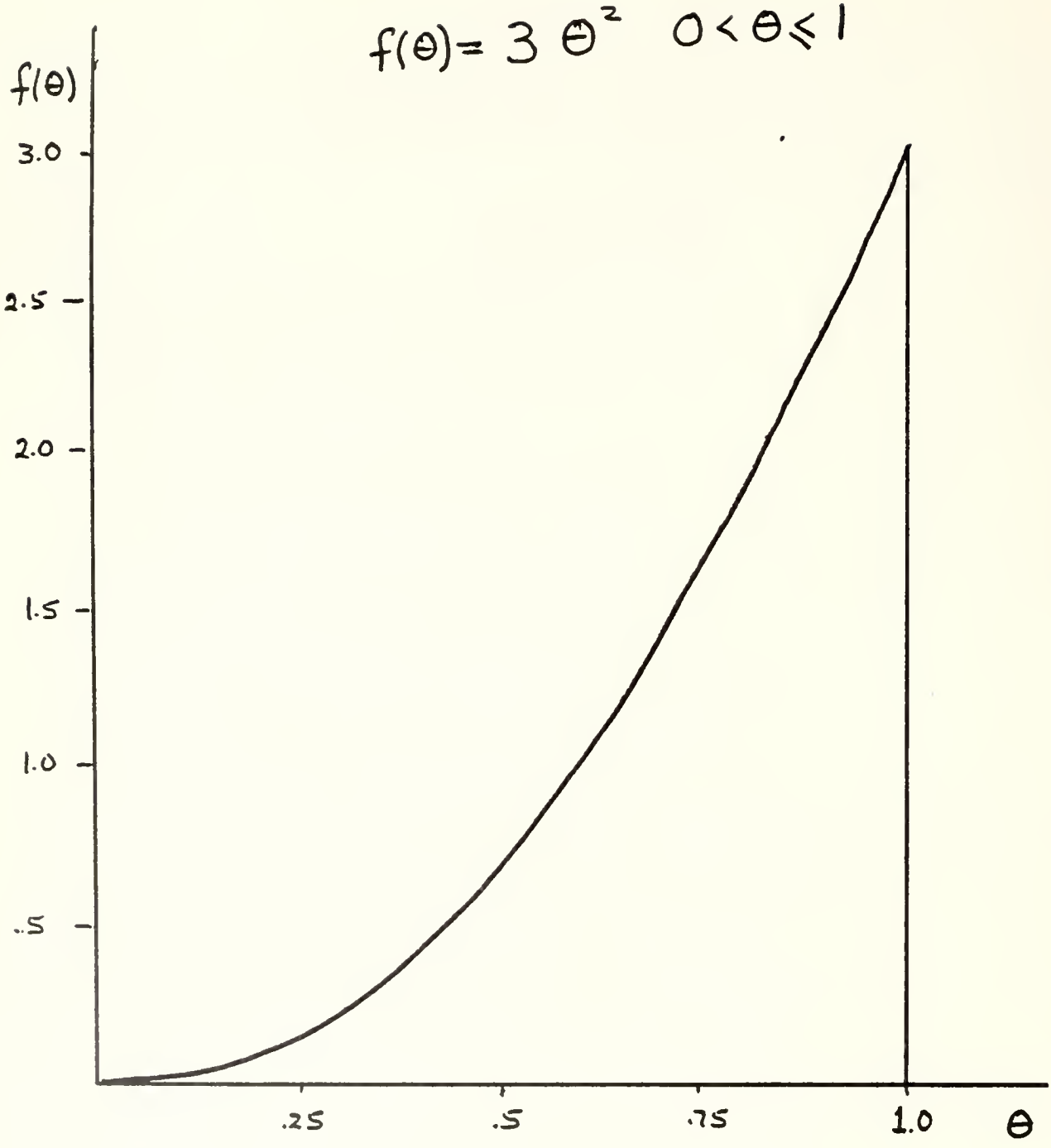

FIGURE 2

Rewriting (10) in the following way allows us to see how much larger the maximum expected variance is

$$E_\theta \mathrm{Var}\underset{\sim}{X} = (1 - \tfrac{M}{Y})\mathrm{Var}Y + \tfrac{M}{Y}(1 - \tfrac{M}{Y})\bar{Y}^2 + \frac{\frac{M}{Y}(1 - \frac{M}{Y})^2}{(2 - \frac{M}{Y})}[\mathrm{Var}Y + \bar{Y}^2]$$

The following table exhibits the values of the factor $\frac{\frac{M}{Y}(1 - \frac{M}{Y})^2}{(2 - \frac{M}{Y})}$ for various values of $\frac{M}{Y}$.

$\frac{M}{Y}$	$\frac{\frac{M}{Y}(1 - \frac{M}{Y})^2}{2 - \frac{M}{Y}}$
.01	.0049
.02	.0097
.03	.0143
.04	.0188
.05	.0231
.06	.0273
.07	.0314
.08	.0353
.09	.0390
.10	.0426

In the previous numerical example, we stated that the smallest expected standard deviation of audited amounts are 16.71 in stratum 1 and 42.83 in stratum 2. With the model just described, the largest expected standard deviations of audited amounts are 17.06 in stratum 1 and 44.83 in stratum 2. Using these maximum amounts, the required sample size increases to 394, with 143 allocated to stratum 1 and 251 to stratum 2.

This numerical example assumes that the estimated audited amount is to be based on a stratified mean estimate of the sample audited amounts. We now consider the situation when the auditor intends to design the sample based on using a stratified difference estimator. From (6), we have

$$E_\theta \underset{\sim}{\text{Var}D} = \pi(\mu_\theta^2 + \sigma_\theta^2)\text{Var}Y + (\pi(1 - \pi)\mu_\theta^2 + \pi\sigma_\theta^2)\bar{Y}^2.$$

Setting $\mu_\theta = \frac{c}{c + 1}$ and $\sigma_\theta^2 = \frac{c}{(c + 1)^2(c + 2)}$, this becomes

$$E_\theta \underset{\sim}{\text{Var}D} = (\pi(\frac{c}{c + 1})^2 + \pi\frac{c}{(c + 1)^2(c + 2)})\text{Var}Y$$

$$+(\pi(1 - \pi)(\frac{c}{c + 1})^2 + \pi\frac{c}{(c + 1)^2(c + 2)})\bar{Y}^2$$

Imposing the condition that $\pi\frac{c}{c + 1} = \frac{M}{\bar{Y}}$, we have

$$(12) \qquad E_\theta \underset{\sim}{\text{Var}D} = [\frac{1}{\pi}(\frac{M}{\bar{Y}})^2 + \frac{\frac{M}{\bar{Y}}(1 - \frac{1}{\pi}\frac{M}{\bar{Y}})^2}{(2 - \frac{1}{\pi}\frac{M}{\bar{Y}})}]\text{Var}Y + [(\frac{M}{\bar{Y}})^2(\frac{1}{\pi} - 1) + \frac{\frac{M}{\bar{Y}}(1 - \frac{1}{\pi}\frac{M}{\bar{Y}})^2}{(2 - \frac{1}{\pi}\frac{M}{\bar{Y}})}]\bar{Y}^2,$$

where π can range from $\frac{M}{\bar{Y}}(c = +\infty)$ to $1(c = \frac{\frac{M}{\bar{Y}}}{1 - \frac{M}{\bar{Y}}})$.
Because $E_\theta \underset{\sim}{\text{Var}D}$ is a strictly decreasing function of π in this range, the maximum value of $E_\theta \underset{\sim}{\text{Var}D}$ occurs at $\pi = \frac{M}{\bar{Y}}(c = +\infty)$, and the largest expected standard deviation is

$$(13) \qquad E_\theta \underset{\sim}{\text{Var}D} = \frac{M}{\bar{Y}}\text{Var}Y + \frac{M}{\bar{Y}}(1 - \frac{M}{\bar{Y}})\bar{Y}^2$$

This result demonstrates that the upper bound cited in (6) is achieved in this class of distribution. Using the same numerical example, the maximum value under this model for the standard deviation of difference amounts in stratum 1 is 11.28, and in stratum 2, 38.40. The required sample size using these amounts is 247 with 76 allocated to stratum 1 and 171 allocated to stratum 2.

As expected, the stratified difference estimator requires a smaller sample size than the stratified mean estimator. In fact, the sample size

required by the stratified difference estimator is less than the sample size required by the stratified mean estimator even when the latter uses the smallest possible variance under the model. However, when compared to the sample size computed using the recorded amounts, the stratified difference estimator requires more. The reason for the increase is that the stratification of the recorded amounts resulted in the top stratum having the square of the coefficient of variation equal to .0131 which is far less than $\frac{M}{Y} = .05$.

Sample Planning. Because both the decision rule and the sample size depend upon the standard error when the monetary error is a material amount (D = M), planning should take this into account. When all monetary errors represent overstatement and the errors are randomly distributed among the population items, the auditor might plan on the basis of the largest possible standard error. Two possible ways of doing this are to maintain the square of the coefficient of variation larger than $\frac{M}{Y}$ within each stratum, or to use an upper bound for the variance of differences within each stratumm.

To implement the first of these, it would be necessary to carry out the stratification process only to the point that the square of the coefficient of variation within each stratum is at least as large as $\frac{M}{Y}$. For the second, the bound displayed in (6) could be used to represent the expected standard deviation of differences within each stratum. Limited empirical tests of this on Neter and Loebbecke's population 4 would suggest that increasing the number of strata beyond 4-6 does not produce any savings in sample size. Table 2 below shows the sample size found using the computer program PLAN2 described in Roberts (1978) for

$\alpha = \beta = .05$, desired precision equal to $295,125 and the population of differences equal to .10.

Number of Strata	Sample Size
2	284
5	194
7	184
9	180
12	178

An additional advantage of planning the sample using either of these procedures is that the auditor can evaluate the sample results. For example, suppose L strata have been used and planning has been based on the maximum standard deviation of differences. In this case, the maximum standard error is as follows

$$\sigma_M^*(\hat{D}_S) = \sqrt{\Sigma N_i(N_i - n_i)\frac{\frac{M}{Y}Var(Y_i) + \frac{M}{Y}(1 - \frac{M}{Y})\bar{Y}_i^2}{n_i}} \quad .$$

The decision rule is to decide there is no material error whenever

$$\hat{D}_S + z_\beta \sigma_M^*(\hat{D}_S) < M.$$

This decision rule may be used regardless of the number of errors found in the sample.

REFERENCES

Cochran, W. G. (1939). The use of analysis of variance in enumeration by sampling. Journal of the American Statistical Association. 34, 492-510.

Cochran, W. G. (1946). Relative accuracy of systematic and stratified random sampling for a certain class of population. Annals of Mathematical Statistics. 17, 164-177.

Deming, W. E. and F. Stephen (1941). On the interpretation of censuses as samples. Journal of the American Statistical Association. 60, 750-771.

Kaplan, R. S. (1973A). A stochastic model for auditing. Journal of Accounting Research. Autumn, 38-46.

Kaplan, R. S. (1973B). Statistical sampling in auditing with auxiliary information estimates. Journal of Accounting Research. Autumn, 238-258.

Madow, W. G. and L. H. Madow (1944). On the theory of systematic sampling. Annals of Mathematical Statistics. 15, 1-24.

Neter, J. and J. K. Loebbecke (1975). Behavior of major statistical estimates in sampling accounting populations. New York: American Institute of Certified Public Accountants.

Roberts, D. M. (1978). Statistical auditing. New York: American Institute of Certified Public Accountants.

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